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Lacunary Wronskians on genus one curves

Greg W. Anderson*

Department of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

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Abstract

Let X be a nonsingular projective curve of genus one defined over an algebraically closed field of characteristic 0. Let D be a divisor of X of degree $n>1$ and let O be a (closed) point of X. As is well known, there exists a unique morphism $\phi_{D,O} : X \to X$ such that $\phi_{D,O}(P)=Q$ if and only if the divisor $nP - D - O + Q$ is principal. Our main result is a simple explicit description of the map $\phi_{D,O}$ in terms of Wronskians and certain Wronskian-like determinants *lacunary* in the sense that derivatives of some orders are skipped. Further, for $n=2,3$ we interpret our main result as a syzygy from classical invariant theory, thus reconciling our work with a circle of ideas treated in two papers by Weil and a recent paper by An, Kim, Marshall, Marshall, McCallum and Perlis.

MSC: 14H40; 14H50; 14H52

Keywords: Jacobians; Genus one; Syzygies

1. Introduction

Let X be a nonsingular projective curve of genus one defined over an algebraically closed field k of characteristic zero. Let D be a divisor of X of degree $n > 1$. Let O be a (closed) point of X. Let $f \mapsto f'$ be a fixed not-identically-vanishing translation-

* Fax: +1 6126262017.

E-mail address: gwanders@math.umn.edu.

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invariant derivation of the function field of X/k and let $f_1, \dots, f_n$ be a k-basis for $H^0(X, \mathcal{O}_X(D))$. As is well known, there exists a unique morphism $\phi_{D,O} : X \to X$ of curves over k such that $\phi_{D,O}(P) = Q$ if and only if the divisor $nP - D - O + Q$ is principal. In this paper we work out a simple explicit description of the map $\phi_{D,O}$ in terms of the Wronskian

$$\begin{vmatrix} f_1^{(0)} & \cdots & f_1^{(n-1)} \\ \vdots & & \vdots \\ f_n^{(0)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

and certain Wronskian-like determinants such as

$$\begin{vmatrix} f_1^{(0)} & \cdots & f_1^{(n-2)} & f_n^{(n)} \\ \vdots & & \vdots & \vdots \\ f_n^{(0)} & \cdots & f_n^{(n-2)} & f_n^{(n)} \end{vmatrix}, \quad \begin{vmatrix} f_1^{(0)} & \cdots & f_1^{(n-3)} & f_1^{(n-1)} & f_1^{(n+1)} \\ \vdots & & \vdots & \vdots & \vdots \\ f_n^{(0)} & \cdots & f_n^{(n-3)} & f_n^{(n-1)} & f_n^{(n+1)} \end{vmatrix}.$$

Determinants of the latter form we call *lacunary* Wronskians because derivatives of some orders are skipped. Our main result is Theorem 2.3.3 below. The proof of the theorem is short, self-contained and more or less elementary. After proving our main result we take pains to reconcile it for $n = 2, 3$ with the results of the papers [Weil 1954,Weil 1983,AKMMMP 2001]; in each of the cited papers the theme developed is that of interpreting syzygies from classical invariant theory as descriptions of maps of the form $\phi_{D,O}$. The paper concludes with various remarks and questions.

2. Lacunary Wronskian identities in a function field of genus one

Throughout this section we fix an algebraically closed field k and a nonsingular projective curve X/k of genus one. At the outset we place no restriction on the characteristic of k. We call elements of k *constants*. When we speak of *points* of X we mean *closed* points.

2.1. Notation and background

2.1.1. General notation. Let K/k be the function field of X/k. Given a divisor D of X, put $L(D) = H^0(X, \mathcal{O}_X(D)) \subset K$. Given a point $P \in X$ and a function $f \in K$, let $\mathrm{ord}_P\, f \in \mathbb{Z} \coprod \{+\infty\}$ be the order of vanishing of f at P. Given a point $P \in X$ and a divisor D of X, let $\mathrm{ord}_P\, D \in \mathbb{Z}$ be the multiplicity with which P appears in D.

2.1.2. Special notation for genus one. Given a divisor D of X of degree 1 and a function $f \in K$, let $f(D) \in k \coprod \{\infty\}$ be the value taken by f at the unique point $P \in X$ such that $D - P$ is a principal divisor. Given a divisor D of degree 0 and a function $f \in K$, let $f_D \in K$ be the unique function such that $f_D(P) = f(P - D)$

for all points $P \in X$. Automorphisms of K/k of the form $f \mapsto f_D$ will be called *translations*. Note that we have

$$\mathrm{ord}_Q\, f_{O-P} = \mathrm{ord}_R\, f$$

for all functions $f \in K$ and points $O, P, Q, R \in X$ such that the divisor $O-P-Q+R$ is principal.

2.1.3. Uniformizers. We say that $t \in K$ is a *uniformizer* at a point $P \in X$ if $\mathrm{ord}_P\, t = 1$. Given points $O, P \in X$ and a uniformizer $s \in K$ at O, note that we have

$$\mathrm{ord}_P\, s_{P-O} = \mathrm{ord}_O\, s = 1$$

and hence the function s_{P-O} is a uniformizer at P.

2.1.4. Lemma. *Let $s, t \in K$ be uniformizers at a point $P \in X$. Fix a function $f \in K$ and consider the Laurent expansions*

$$f = \sum_{i\in\mathbb{Z}} a_i s^i = \sum_{i\in\mathbb{Z}} b_i t^i \quad (a_i, b_i \in k, \quad a_i = 0 = b_i \text{ for } i \ll 0)$$

of f at P in powers of s and of t, respectively. We have

$$i \leqslant -2 + \mathrm{ord}_P\, f + \mathrm{ord}_P(s-t) \Rightarrow a_i = b_i$$

for all $i \in \mathbb{Z}$.

Proof. It is enough to observe that

$$\mathrm{ord}_P(s^i - t^i) \geqslant -1 + i + \mathrm{ord}_P(s-t)$$

for all $i \in \mathbb{Z}$. □

2.2. Laurent series calculations

2.2.1. The coefficients $c_i(D, P, t)$. Suppose we are given the following:

- A divisor D of X.
- A point $P \in X$ such that the divisor $(\deg D) \cdot P - D$ is nonprincipal.
- A uniformizer $t \in K$ at P.

By Riemann–Roch there exists for each positive integer v a unique function

$$f_v \in L(v \cdot P - (\deg D) \cdot P + D)$$

such that

$$\mathrm{ord}_P \left(t^{-v+\deg D-\mathrm{ord}_P D} - f_v\right) \geqslant \deg D - \mathrm{ord}_P D.$$

By considering the Laurent expansion

$$f_2/f_1 = t^{-1} + \sum_{i=0}^{\infty} c_i(D, P, t)t^i$$

at P in powers of t we define a coefficient

$$c_i(D, P, t) \in k$$

for each nonnegative integer i.

2.2.2. Lemma. *Let D, P, and t be as above. Fix an integer $i \geqslant 0$. Fix distinct points $O, Q \in X$. Let $R \in X$ be the unique point such that the divisor $O - P - Q + R$ is principal. Let $s \in K$ be another uniformizer at P. The following hold*:

(1) $c_i(D, P, t) = c_i(D + P, P, t)$.
(2) *If $i > 0$, then $c_i(D, P, t)$ depends only on the divisor class of D.*
(3) $c_i(-R, P, t) = c_i(-Q, O, t_{O-P})$.
(4) *If $i \leqslant -3 + \mathrm{ord}_P(s - t)$, then $c_i(D, P, s) = c_i(D, P, t)$.*

Proof. Just during the course of this proof let the function f_v figuring in the definition of the coefficient $c_i(D, P, t)$ be denoted by $f_v[D, P, t]$.

1. We have $f_v[D + P, P, t] = f_v[D, P, t]$.
2. Let D' be a divisor linearly equivalent to D. Let $g \in K$ be a function with divisor $D - D'$. We have

$$g \cdot L(v \cdot P - (\deg D) \cdot P + D) = L(v \cdot P - (\deg D') \cdot P + D')$$

and hence for some constants $c \neq 0$ and c' we have

$$gf_1[D, P, t] = cf_1[D', P, t], \quad gf_2[D, P, t] = cf_2[D', P, t] + c' f_1[D', P, t].$$

3. We have $f_v[-R, P, t]_{O-P} = f_v[-Q, O, t_{O-P}]$.
4. By hypothesis $\mathrm{ord}_P(s - t) \geqslant 3$, hence for $v = 1, 2$ we have

$$\mathrm{ord}_P \left(s^{-v+\deg D - \mathrm{ord}_P D} - t^{-v+\deg D - \mathrm{ord}_P D} \right) \geqslant \deg D - \mathrm{ord}_P D$$

and hence $f_v[D, P, s] = f_v[D, P, t]$. The result now follows by Lemma 2.1.4. $\square$

2.2.3. Proposition. *Fix points $O, P \in X$. Let $x, y \in K$ be functions regular away from O such that*

$$\mathrm{ord}_O\, x = -2, \quad \mathrm{ord}_O\, y = -3, \quad \mathrm{ord}_O(y^2 - x^3) > -6.$$

Let $a \in k$ be defined by the condition

$$\mathrm{ord}_O(y^2 + axy - x^3) > -5.$$

Put

$$t = (x/y)_{P-O},$$

thereby defining a uniformizer at P. We have

$$\begin{aligned} c_1(D, P, t) &= x((\deg\ D) \cdot P - D + O), \\ c_2(D, P, t) &= -(y + ax)((\deg\ D) \cdot P - D + O), \end{aligned}$$

for all divisors D such that the divisor $(\deg\ D) \cdot P - D$ *is nonprincipal.*

Proof. In the identities we seek to prove the right sides depend only on the divisor class of $(\deg D) \cdot P - D$, and so do the left sides by 1 and 2 of Lemma 2.2.2. Therefore we may assume without loss of generality that

$$D = -R$$

for some point $R \in X$ distinct from P. Let $Q \in X$ be the unique point such that the divisor

$$(\deg\ D) \cdot P - D + O - Q = O - P - Q + R$$

is principal. Note that since $P \neq R$, we also have $O \neq Q$, and hence both x and y are regular at Q. Further, by the definitions we have

$$x((\deg\ D) \cdot P - D + O) = x(Q), \quad y((\deg\ D) \cdot P - D + O) = y(Q).$$

Now put

$$s = x/y,$$

thereby defining a uniformizer at O. By 3 of Lemma 2.2.2 we have

$$c_i(D, P, t) = c_i(-R, P, t) = c_i(-Q, O, t_{O-P}) = c_i(-Q, O, s)$$

for all integers $i \geqslant 0$. It remains only to calculate $c_i(-Q, O, s)$ for $i = 1, 2$. To this end consider the functions

$$f_1 = x - x(Q) \in K, \quad f_2 = y - y(Q) - af_1 \in K.$$

By hypothesis we have Laurent expansions

$$x = s^{-2} + as^{-1} + \cdots, \quad y = s^{-3} + as^{-2} + \cdots$$

at O in powers of s, and moreover, as noted above, x and y are regular at Q. Therefore for $v = 1, 2$ we have

$$f_v \in L(v \cdot O + O - Q), \quad \mathrm{ord}_O(s^{-v-1} - f_v) \geqslant -1.$$

Clearly we have a Laurent expansion

$$\begin{aligned} f_2/f_1 &= s^{-1}\frac{1 - y(Q)/y}{1 - x(Q)/x} - a \\ &= s^{-1} - a + x(Q)s - (y(Q) + ax(Q))s^2 + \cdots \end{aligned}$$

at O in powers of s. Finally, by the definitions we have

$$c_1(-Q, O, s) = x(Q), \quad c_2(-Q, O, s) = -(y(Q) + ax(Q))$$

and we are done. $\square$

2.2.4. Proposition. *Fix a divisor D of X of degree $n > 1$, a point $P \in X$ such that the divisor $n \cdot P - D$ is nonprincipal and a uniformizer $t \in K$ at P. Let $h_1, \ldots, h_n$ be any k-basis for $L(D)$. Assemble the coefficients of the Taylor expansions*

$$t_P^{\mathrm{ord}_P D} h_i = \sum_{j=0}^{\infty} A_{ij} t^j$$

at P in powers of t into a matrix A with rows indexed by $1, \ldots, n$ and columns indexed by the nonnegative integers. Let W be the determinant of the leftmost n by n block of A. For

$$(\alpha, \beta) \in \{0, 1, 2, \ldots\} \times \{0, \ldots, n-1\},$$

let $W_{\alpha\beta}$ be the determinant of the square matrix obtained by striking all columns of A with indices not in the set

$$\{0, \ldots, n-1\} \setminus \{n - 1 - \beta\} \cup \{\alpha + n\}.$$

Then W does not vanish and we have

$$\begin{aligned} c_1(D, P, t) &= -\frac{W_{10}}{W} + \left(\frac{W_{00}}{W}\right)^2 - \frac{W_{01}}{W}, \\ c_2(D, P, t) &= -\frac{W_{20}}{W} + \frac{W_{00}W_{01}}{W^2} - \left(\frac{W_{00}}{W}\right)^3 + \frac{2W_{10}W_{00}}{W^2} - \frac{W_{11}}{W}. \end{aligned}$$

Proof. We have

$$W = \begin{vmatrix} A_{10} & \cdots & A_{1,n-1} \\ \vdots & & \vdots \\ A_{n0} & \cdots & A_{n,n-1} \end{vmatrix},$$

$$W_{00} = \begin{vmatrix} A_{10} & \cdots & A_{1,n-2} & A_{1,n} \\ \vdots & & \vdots & \vdots \\ A_{n0} & \cdots & A_{n,n-2} & A_{n,n} \end{vmatrix},$$

$$W_{10} = \begin{vmatrix} A_{10} & \cdots & A_{1,n-2} & A_{1,n+1} \\ \vdots & & \vdots & \vdots \\ A_{n0} & \cdots & A_{n,n-2} & A_{n,n+1} \end{vmatrix},$$

$$W_{20} = \begin{vmatrix} A_{10} & \cdots & A_{1,n-2} & A_{1,n+2} \\ \vdots & & \vdots & \vdots \\ A_{n0} & \cdots & A_{n,n-2} & A_{n,n+2} \end{vmatrix},$$

$$W_{01} = \begin{vmatrix} A_{10} & \cdots & A_{1,n-3} & A_{1,n-1} & A_{1n} \\ \vdots & & \vdots & \vdots & \vdots \\ A_{n0} & \cdots & A_{n,n-3} & A_{n,n-1} & A_{nn} \end{vmatrix},$$

$$W_{11} = \begin{vmatrix} A_{10} & \cdots & A_{1,n-3} & A_{1,n-1} & A_{1,n+1} \\ \vdots & & \vdots & \vdots & \vdots \\ A_{n0} & \cdots & A_{n,n-3} & A_{n,n-1} & A_{n,n+1} \end{vmatrix},$$

and so on. Just as in the definition of the coefficient $c_i(D, P, t)$, for each positive integer v let

$$f_v \in L(v \cdot P - nP + D)$$

be the unique function such that

$$\mathrm{ord}_P \left(t^{-v+n-\mathrm{ord}_P\, D} - f_v \right) \geqslant n - \mathrm{ord}_P\, D.$$

Now we are free to replace $h_1, \dots, h_n$ by any k-basis of $L(D)$ since by doing so we merely multiply all the determinants W and $W_{\alpha\beta}$ by a common nonzero constant factor. We may therefore assume without loss of generality that

$$h_v = f_{n-v+1}$$

for $v = 1, \ldots, n$, in which case we have

$$\begin{bmatrix} A_{10} & \cdots & A_{1,n-1} \\ \vdots & & \vdots \\ A_{n0} & \cdots & A_{n,n-1} \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}, \quad A_{n-\beta,\alpha+n} = (-1)^{\beta} \frac{W_{\alpha\beta}}{W}.$$

It is possible now to obtain the claimed identities by expanding the ratio

$$f_2/f_1 = \frac{t^{-1} - \frac{W_{01}}{W} t - \frac{W_{11}}{W} t^2 + \cdots}{1 + \frac{W_{00}}{W} t + \frac{W_{10}}{W} t^2 + \frac{W_{20}}{W} t^3 + \cdots}$$

at P in powers of t. We omit the tedious but straightforward calculation needed to complete the proof. $\square$

2.3. Lacunary Wronskian identities in characteristic 0

2.3.1. The revised setting. We assume hereafter that the constant field k is of characteristic zero. We fix a not-identically-vanishing derivation $f \mapsto f'$ of the function field K/k commuting with all translations. Let $f^{(i)}$ denote the result of i times differentiating f.

2.3.2. Lemma. *Fix a positive integer N and a point $O \in X$ arbitrarily. There exists a uniformizer $t \in K$ at O such that*

$$\mathrm{ord}_O(t' - 1) \geqslant N - 1.$$

Moreover, for any uniformizer $t \in K$ at O satisfying the condition above, point $P \in X$ and function $f \in K$ regular at P, we have

$$\mathrm{ord}_P \left(f - \sum_{n=0}^{N-1} \frac{f^{(i)}(P)}{i!} t_{P-O}^{i} \right) \geqslant N.$$

Proof. Freshman calculus. $\square$

2.3.3. Theorem. *Fix a point $O \in X$ arbitrarily. Let $\wp \in K$ be the unique function on X regular away from O such that*

$$\mathrm{ord}_O(\wp) = -2, \quad \mathrm{ord}_O((\wp')^2 - 4\wp^3) \geqslant -2.$$

Let $g_2, g_3 \in k$ be the unique constants such that the Weierstrass differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

holds. Fix a divisor D of X of degree $n > 1$ and a k-basis $h_1, \ldots, h_n$ for $L(D)$. Let A be the matrix with entries $A_{ij} = h_i^{(j)}/j! \in K$ where $i = 1, \ldots, n$ and j ranges over the nonnegative integers. Let $W \in K$ be the determinant of the leftmost n-by-n block of A. For

$$(\alpha, \beta) \in \{0, 1, 2, \ldots\} \times \{0, \ldots, n-1\}$$

let $W_{\alpha\beta} \in K$ be the determinant of the square matrix obtained from A by striking all columns with indices not in the set

$$\{0, 1, \ldots, n-1\} \setminus \{n-1-\beta\} \cup \{n+\alpha\}.$$

(*We call determinants of the form $W_{\alpha\beta}$ lacunary Wronskians.*) *Put*

$$G = -\frac{W_{10}}{W} + \left(\frac{W_{00}}{W}\right)^2 - \frac{W_{01}}{W} \in K,$$

$$H = -\frac{W_{20}}{W} + \frac{W_{00}W_{01}}{W^2} - \left(\frac{W_{00}}{W}\right)^3 + \frac{2W_{10}W_{00}}{W^2} - \frac{W_{11}}{W} \in K.$$

(i) *The determinant W does not vanish identically, hence the functions G and H are well-defined.* (ii) *We have*

$$\wp(n \cdot P - D + O) = G(P), \quad \wp'(n \cdot P - D + O) = 2H(P)$$

for all points $P \in X$. (iii) *Consequently the identities*

$$4H^2 = 4G^3 - g_2G - g_3, \quad G' = 2nH$$

hold.

Proof. Only (i) and (ii) require proof. Arbitrarily fix a point $P \in X$ such that $n \cdot P - D$ is nonprincipal and $\mathrm{ord}_P\, D = 0$. It suffices to prove that $W(P) \neq 0$ and that (ii) holds for this particular point P. By Lemma 2.3.2 there exists a uniformizer $t \in K$ at O such that

$$\mathrm{ord}_O(t' - 1) \geqslant \max(4, n+2),$$

and we have

$$\mathrm{ord}_P\left(h_i - \sum_{j=0}^{n+2} A_{ij}(P) \cdot t_{P-O}^j\right) \geqslant n+3$$

for $i = 1, \ldots, n$. By Proposition 2.2.4 we indeed have $W(P) \neq 0$, and consequently both G and H are regular at P. By the cited proposition we further have

$$G(P) = c_1(D, P, t_{P-O}), \quad H(P) = c_2(D, P, t_{P-O}).$$

Now put

$$x = \wp, \quad y = -\wp'/2, \quad s = x/y.$$

We have

$$s' - 1 = \left(\frac{g_2}{2} \cdot x + \frac{3g_3}{4}\right) \Big/ y^2, \quad \mathrm{ord}_O(s' - 1) \geqslant 4,$$

hence

$$\mathrm{ord}_P(s_{P-O} - t_{P-O}) = \mathrm{ord}_O(s - t) \geqslant 5$$

by Lemma 2.3.2 and hence

$$c_1(D, P, t_{P-O}) = c_1(D, P, s_{P-O}), \quad c_2(D, P, t_{P-O}) = c_2(D, P, s_{P-O})$$

by 4 of Lemma 2.2.2. Finally, since

$$\mathrm{ord}_O(y^2 - x^3) \geqslant -2 > -5,$$

we have

$$\begin{aligned} c_1(D, P, s_{P-O}) &= x(n \cdot P - D + O) = \wp(n \cdot P - D + O), \\ c_2(D, P, s_{P-O}) &= -y(n \cdot P - D + O) = \wp'(n \cdot P - D + O)/2 \end{aligned}$$

by Proposition 2.2.4. We are done. □

2.3.4. Remark. With D and O as in the theorem, let $\phi_{D,O} : X \to X$ be the map defined in the Introduction. Then (ii) of the theorem can be rewritten in the form

$$G = \phi^*_{D,O}\wp, \quad 2H = \phi^*_{D,O}\wp'.$$

Thus, as promised in the Introduction, Theorem 2.3.3 provides a simple explicit description of the map $\phi_{D,O}$ in terms of lacunary Wronskians.

3. Interpretation in terms of classical invariant theory

In order to reconcile Theorem 2.3.3 to the work of [Weil 1954,Weil 1983,AKMMMP 2001], as well as to explain to the reader how Theorem 2.3.3 works out in practice, we take a close look at the theorem in the special cases $n = 2$ and 3, in each case coming up with an interpretation of the lacunary Wronskian identity

$$4H^2 = 4G^3 - g_2 G - g_3$$

as a syzygy from classical invariant theory. We continue working with the notation and in the setting of Theorem 2.3.3.

3.1. The case $n = 2$

3.1.1. Specialization of the setting. Assume that X/k is the smooth projective model of the affine plane curve

$$y^2 = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4,$$

where $a_0, a_1, a_2, a_3, a_4 \in k$ are algebraically independent over $\mathbb{Q}$. Assume further that the given translation-invariant derivation $f \mapsto f'$ of the function field K/k is dual to the nonzero differential dx/y of the first kind on X. Then the formulas

$$x' = y, \quad y' = 2a_0x^3 + 6a_1x^2 + 6a_2x + 2a_3$$

and the rules of freshman calculus uniquely determine the given derivation $f \mapsto f'$.

3.1.2. Calculation of $\wp$. Fix a square root α of a_0 in k. There exist unique points $O, P \in X$ such that

$$\mathrm{ord}_O(x) = -1, \quad \mathrm{ord}_O(y) = -2, \quad \mathrm{ord}_O(y - \alpha x^2) \geqslant -1,$$

and

$$\mathrm{ord}_P(x) = -1, \quad \mathrm{ord}_P(y) = -2, \quad \mathrm{ord}_P(y + \alpha x^2) \geqslant -1,$$

respectively. Put

$$\begin{aligned}
\wp &= \alpha y/2 + a_0x^2/2 + a_1x + a_2/2, \\
i &= a_0a_4 - 4a_1a_3 + 3a_2^2, \\
j &= a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3.
\end{aligned}$$

The function $\wp$ has a double pole at the point O and no other singularity. Moreover, the function $\wp$ satisfies the Weierstrass differential equation

$$(\wp')^2 = 4\wp^3 - i\wp - j.$$

These properties of $\wp$ are easy to verify with a computer algebra system.

3.1.3. Calculation of G and H. Put

$$D := O + P,$$

thereby defining an effective divisor of X of degree 2. Take the functions 1 and x as a k-basis for $L(D)$. According to the recipe of Theorem 2.3.3 we have

$$W = \begin{vmatrix} 1 & 1' \\ x & x' \end{vmatrix} = x' = y,$$

further and similarly we have

$$W_{00} = \frac{x''}{2}, \quad W_{10} = \frac{x'''}{6}, \quad W_{20} = \frac{x''''}{24}, \quad W_{01} = 0 = W_{11},$$

and hence we have

$$G = -\frac{1}{6}\frac{x'''}{x'} + \frac{1}{4}\left(\frac{x''}{x'}\right)^2, \quad H = -\frac{1}{24}\frac{x''''}{x'} - \frac{1}{8}\left(\frac{x''}{x'}\right)^3 + \frac{1}{6}\frac{x'''x''}{(x')^2}.$$

According to Theorem 2.3.3 we must have

$$4H^2 = 4G^3 - iG - j, \quad G' = 4H.$$

The latter identities are easy to double check with a computer algebra system.

3.1.4. Connection with classical invariant theory. Consider the binary biquadratic form

$$U(\xi, \eta) = a_0\xi^4 + 4a_1\xi^3\eta + 6a_2\xi^2\eta^2 + 4a_3\xi\eta^3 + a_4\eta^4$$

in independent variables ξ and η with coefficients in k. Put

$$g = g(\xi, \eta) = -\frac{1}{144}\begin{vmatrix} U_{\xi\xi} & U_{\xi\eta} \\ U_{\eta\xi} & U_{\eta\eta} \end{vmatrix}, \quad h = h(\xi, \eta) = \frac{1}{8}\begin{vmatrix} U_{\xi} & U_{\eta} \\ g_{\xi} & g_{\eta} \end{vmatrix},$$

thereby associating to U forms of degree 4 and 6, respectively. The form U, its covariants g and h, and its invariants i and j satisfy a well known syzygy

$$h^2 = 4g^3 - igU^2 - jU^3,$$

cf. [Weil 1954,Weil 1983] or [AKMMMP 2001, p. 307]. It is not hard to verify the relations

$$g(x, 1) = y^2 G, \quad h(x, 1) = -2y^3 H$$

with the help of a computer algebra system. Thus we reconcile Theorem 2.3.3 with [Weil 1954,Weil 1983,AKMMMP 2001].

3.2. The case $n = 3$

3.2.1. Specialization of the setting. Consider the ternary cubic form

$$U = U(\xi, \eta, \zeta) = a\xi^3 + b\eta^3 + c\zeta^3 + 3a_2\xi^2\eta + 3a_3\xi^2\zeta + 3b_1\eta^2\xi + 3b_3\eta^2\zeta$$
$$+3c_1\zeta^2\xi + 3c_2\zeta^2\eta + 6m\xi\eta\zeta$$

in independent variables ξ, η and ζ, where the coefficients $a, \ldots, m \in k$ are algebraically independent over $\mathbb{Q}$. We suppose now that X/k is the nonsingular projective curve over k defined by the equation $U = 0$. Let x and y denote the meromorphic functions on X represented by the ratios ξ/ζ and η/ζ, respectively. We assume that the given translation-invariant derivation of the function field K/k is dual to the differential

$$\frac{dx}{U_\eta(x, y, 1)} = -\frac{dy}{U_\xi(x, y, 1)}$$

of the first kind on X, in which case the formulas

$$x' = U_\eta(x, y, 1), \quad y' = -U_\xi(x, y, 1)$$

and the rules of sophomore calculus uniquely determine the given derivation $f \mapsto f'$.

3.2.2. Calculation of $\wp$. Factor the binary cubic form $U(\xi, \eta, 0)$ over k thus:

$$U(\xi, \eta, 0) = b(\eta - r_1\xi)(\eta - r_2\xi)(\eta - r_3\xi).$$

There exist unique and distinct points $P_i \in X$ for $i = 1, 2, 3$ such that

$$\mathrm{ord}_{P_i}(x) = \mathrm{ord}_{P_i}(y) = -1, \quad \mathrm{ord}_{P_i}(y - r_i x) \geqslant 0$$

for $i = 1, 2, 3$. Put

$$\begin{aligned}
\wp \;=\; & (b^2r_2^2r_3^2 + b^2r_1^2r_2r_3 - b^2r_1r_2^2r_3 - b^2r_1r_2r_3^2)x^2 \\
& +(-b^2r_1^2r_2 - b^2r_1^2r_3 + 2b^2r_1r_2r_3 - b^2r_2r_3^2 - b^2r_2^2r_3 + b^2r_1r_3^2 + b^2r_1r_2^2)xy \\
& +(-b^2r_1r_2 - b^2r_1r_3 + b^2r_2r_3 + b^2r_1^2)y^2 \\
& +(3a_3br_2 + 3a_3br_3 + 3bb_3r_1r_2r_3 - 3a_3br_1 + 6bmr_2r_3)x \\
& +(-6bmr_1 - 3a_3b - 3bb_3r_1r_3 - 3bb_3r_1r_2 + 3bb_3r_2r_3)y \\
& +3m^2 - bc_2r_1r_3 - bc_2r_1r_2 - 2bc_1r_1 + bc_1r_2 + 2bc_2r_2r_3 - 3a_3b_3 + bc_1r_3,
\end{aligned}$$

thereby defining a meromorphic function on X regular away from the points P_1, P_2 and P_3. With the help of a computer algebra system it is not hard to verify that $\wp$ has no singularity other than a double pole at P_1 and satisfies the Weierstrass differential equation

$$(\wp')^2 = 4\wp^3 + 108S\wp - 27T,$$

where S and T are the classically known invariants of the ternary cubic form U written down in [AKMMMP 2001, pp. 309–310].

3.2.3. Calculation of G and H. To simplify writing put

$$f^{[i]} = f^{(i)}/i!$$

for all $f \in K$ and nonnegative integers i. Put

$$D := P_1 + P_2 + P_3,$$

thereby defining an effective divisor of X of degree 3. (Equivalently, D is the divisor defined by the equation $\zeta = 0$.) Take the functions 1, x and y as a k-basis for $L(D)$. Following the recipe of Theorem 2.3.3, we have

$$W = \begin{vmatrix} 1^{[0]} & 1^{[1]} & 1^{[2]} \\ x^{[0]} & x^{[1]} & x^{[2]} \\ y^{[0]} & y^{[1]} & y^{[2]} \end{vmatrix} = \begin{vmatrix} x^{[1]} & x^{[2]} \\ y^{[1]} & y^{[2]} \end{vmatrix},$$

further and similarly we have

$$W_{00} = \begin{vmatrix} x^{[1]} & x^{[3]} \\ y^{[1]} & y^{[3]} \end{vmatrix}, \quad W_{10} = \begin{vmatrix} x^{[1]} & x^{[4]} \\ y^{[1]} & y^{[4]} \end{vmatrix}, \quad W_{20} = \begin{vmatrix} x^{[1]} & x^{[5]} \\ y^{[1]} & y^{[5]} \end{vmatrix},$$

$$W_{01} = \begin{vmatrix} x^{[2]} & x^{[3]} \\ y^{[2]} & y^{[3]} \end{vmatrix}, \quad W_{11} = \begin{vmatrix} x^{[2]} & x^{[4]} \\ y^{[2]} & y^{[4]} \end{vmatrix},$$

and hence we have

$$G = -\frac{\begin{vmatrix} x^{[1]} & x^{[4]} \\ y^{[1]} & y^{[4]} \end{vmatrix}}{\begin{vmatrix} x^{[1]} & x^{[2]} \\ y^{[1]} & y^{[2]} \end{vmatrix}} + \left(\frac{\begin{vmatrix} x^{[1]} & x^{[3]} \\ y^{[1]} & y^{[3]} \end{vmatrix}}{\begin{vmatrix} x^{[1]} & x^{[2]} \\ y^{[1]} & y^{[2]} \end{vmatrix}} \right)^2 - \frac{\begin{vmatrix} x^{[2]} & x^{[3]} \\ y^{[2]} & y^{[3]} \end{vmatrix}}{\begin{vmatrix} x^{[1]} & x^{[2]} \\ y^{[1]} & y^{[2]} \end{vmatrix}},$$

$$H = -\frac{\begin{vmatrix} x^{[1]} & x^{[5]} \\ y^{[1]} & y^{[5]} \end{vmatrix}}{\begin{vmatrix} x^{[1]} & x^{[2]} \\ y^{[1]} & y^{[2]} \end{vmatrix}} + \frac{\begin{vmatrix} x^{[1]} & x^{[3]} \\ y^{[1]} & y^{[3]} \end{vmatrix} \cdot \begin{vmatrix} x^{[2]} & x^{[3]} \\ y^{[2]} & y^{[3]} \end{vmatrix}}{\begin{vmatrix} x^{[1]} & x^{[2]} \\ y^{[1]} & y^{[2]} \end{vmatrix}^2} - \left(\frac{\begin{vmatrix} x^{[1]} & x^{[3]} \\ y^{[1]} & y^{[3]} \end{vmatrix}}{\begin{vmatrix} x^{[1]} & x^{[2]} \\ y^{[1]} & y^{[2]} \end{vmatrix}} \right)^3$$

$$+2 \frac{\begin{vmatrix} x^{[1]} & x^{[4]} \\ y^{[1]} & y^{[4]} \end{vmatrix} \cdot \begin{vmatrix} x^{[1]} & x^{[3]} \\ y^{[1]} & y^{[3]} \end{vmatrix}}{\begin{vmatrix} x^{[1]} & x^{[2]} \\ y^{[1]} & y^{[2]} \end{vmatrix}^2} - \frac{\begin{vmatrix} x^{[2]} & x^{[4]} \\ y^{[2]} & y^{[4]} \end{vmatrix}}{\begin{vmatrix} x^{[1]} & x^{[2]} \\ y^{[1]} & y^{[2]} \end{vmatrix}}.$$

According to Theorem 2.3.3 we must have

$$4H^2 = 4G^3 + 108SG - 27T, \quad G' = 6H.$$

Now unlike, say, the identities of Section 3.1.3, which on a run-of-the-mill laptop computer running off-the-shelf algebra software take only seconds of computer time to double check, the ones above turn out to be lot more difficult to double check. Indeed, we tried hard to double check them, but not being patient enough to wait for the computer to finish or crash, we failed. So then we tried randomly specializing the coefficients $a, \ldots, m$ to integers in the range $[-10^6, 10^6]$, and fortunately that tactic worked: thus specialized the identities above took only tens of seconds of computer time to double check.

3.2.4. Connection with classical invariant theory. Put

$$V = \frac{1}{216} \begin{vmatrix} U_{\xi\xi} & U_{\xi\eta} & U_{\xi\zeta} \\ U_{\eta\xi} & U_{\eta\eta} & U_{\eta\zeta} \\ U_{\zeta\xi} & U_{\zeta\eta} & U_{\zeta\zeta} \end{vmatrix},$$

$$\Theta = -\frac{1}{144} \operatorname{trace} \left(\begin{bmatrix} 0 & \zeta & -\eta \\ -\zeta & 0 & \xi \\ \eta & -\xi & 0 \end{bmatrix} \operatorname{adj} \left(\begin{bmatrix} U_{\xi\xi} & U_{\xi\eta} & U_{\xi\zeta} \\ U_{\eta\xi} & U_{\eta\eta} & U_{\eta\zeta} \\ U_{\zeta\xi} & U_{\zeta\eta} & U_{\zeta\zeta} \end{bmatrix} \right) \right.$$
$$\left. \times \begin{bmatrix} 0 & \zeta & -\eta \\ -\zeta & 0 & \xi \\ \eta & -\xi & 0 \end{bmatrix} \operatorname{adj} \left(\begin{bmatrix} V_{\xi\xi} & V_{\xi\eta} & V_{\xi\zeta} \\ V_{\eta\xi} & V_{\eta\eta} & V_{\eta\zeta} \\ V_{\zeta\xi} & V_{\zeta\eta} & V_{\zeta\zeta} \end{bmatrix} \right) \right),$$

$$J = -\frac{1}{9} \begin{vmatrix} U_\xi & V_\xi & \Theta_\xi \\ U_\eta & V_\eta & \Theta_\eta \\ U_\zeta & V_\zeta & \Theta_\zeta \end{vmatrix} = -\frac{1}{9} \begin{vmatrix} U_\xi & V_\xi & \Theta_\xi \\ U_\eta & V_\eta & \Theta_\eta \\ 3U/\zeta & 3V/\zeta & 6\Theta/\zeta \end{vmatrix},$$

thereby naturally associating to U homogeneous forms

$$V = V(\xi, \eta, \zeta), \quad \Theta = \Theta(\xi, \eta, \zeta), \quad J = J(\xi, \eta, \zeta)$$

of degrees 3, 6 and 9, respectively. The form U, its covariants V, Θ and J, and its invariants S and T satisfy the syzygy

$$J^2 = 4\Theta^3 + 108 S\Theta V^4 - 27TV^6 \bmod U.$$

We copied this syzygy out of [AKMMMP 2001, p. 310]; see the cited paper for the original (awe-inspiring!) 19th century references. It is not too hard to check the identities

$$W = 27V(x, y, 1), \quad W^2 G = 27^2 \Theta(x, y, 1)$$

with a computer algebra system; we only needed a couple dozen lines of code and the computations took only a few minutes. We further claim that

$$2W^3 H = -27^3 J(x, y, 1).$$

The claim granted, the task of reconciling Theorem 2.3.3 to the work of [AKMMMP 2001] in the case $n = 3$ is finished.

We turn to the proof of the claim. We are forced to give a direct proof because the claim is resistant to verification by brute force. Fortunately the proof is easy. On the one hand, we have

$$18W^3 H = 3(W^2 G)' W - 6W'(W^2 G)$$

by (iii) of Theorem 2.3.3. On the other hand, we have

$$\begin{aligned} -9J(x, y, 1) &= \begin{vmatrix} U_\xi(x, y, 1) & V_\xi(x, y, 1) & \Theta_\xi(x, y, 1) \\ U_\eta(x, y, 1) & V_\eta(x, y, 1) & \Theta_\eta(x, y, 1) \\ 0 & 3V(x, y, 1) & 6\Theta(x, y, 1) \end{vmatrix} \\ &= 3\Theta(x, y, 1)' V(x, y, 1) - 6V(x, y, 1)'\Theta(x, y, 1) \end{aligned}$$

by the definitions. Comparison of what we have already proved to the two "hands" proves the claim.

4. Concluding remarks and questions

4.1. Extension to positive characteristic. Since the theory of the coefficients $c_i(D, P, t)$ developed in Section 2.2 of this paper is valid in all characteristics, it may be possible to devise a version of Theorem 2.3.3 valid in all characteristics.

4.2. The cases $n = 4, 5$. Following the pattern set in Section 3 of this paper, it may be possible to reconcile Theorem 2.3.3 in the case $n = 4$ to the work of [AKMMMP 2001] concerning intersections of quadrics in 3-space. But calculations in the case $n = 4$ as naive and brutal as those we undertook above in the case $n = 3$ are sure to be much more involved. Recently, [Fisher 2002] obtained results for $n = 5$ in the same spirit as those of [AKMMMP 2001] and these results we think must be well-nigh impossible to reconcile with ours by brutal calculation. Clearly, new ideas are needed to connect Theorem 2.3.3 with classical invariant theory in a simple way.

4.3. An ill-posed problem. For $n \geqslant 6$ the lacunary Wronskian identity $4H^2 = 4G^3 - g_2G - g_3$ may be a syzygy in search of an invariant-theoretic interpretation. What might that interpretation be? Perhaps the author's recent work [Anderson 2002] on elementary construction of Jacobians gives some clues.

4.4. Fermionic Fock space. The author discovered Theorem 2.3.3 in the course of a still-continuing investigation of number-theoretic applications of fermionic Fock space. The elementary proof of the theorem given in this paper was found later. But our original point of view might still be useful for discovering higher-genus generalizations of Theorem 2.3.3. For an introduction to the fermionic Fock space technique see the author's paper [Anderson 2004].

4.5. Relations with the work of O'Neil. In [O'Neil 2002] Jacobians of genus one curves are constructed by an elegant procedure yielding elliptic curves not in Weierstrass normal form but rather in the same form as the given genus one curve. It is an interesting problem to find an analogue of Theorem 2.3.3 in O'Neil's set up.

4.6. Explicit n-descents. Theorem 2.3.3 may be useful for implementing explicit n-descents on elliptic curves over number fields. This line of thought is strongly suggested by the papers [Weil 1954, Weil 1983, AKMMMP 2001, Fisher 2002, O'Neil 2002], and the references cited in those papers.

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